TAO, Vol. 11, No. 2, 423-438, June 2000

Magnetospheric Configuration with Magnetotail Current

Tyan Yeh[1,*]

(Manuscript received 17 June 1999, in final form 24 November 1999)

ABSTRACT

Magnetospheric configuration with magnetotail current is studied on a two-dimensional formulation. The planar magnetic field is represented by an analytic function of a complex variable that represents points of the noon-midnight plane. In our treatment, which accounts for both the Chapman-Ferraro magnetopause current and the magnetotail current, the function that describes the magnetospheric magnetic field is a rational function. Thus, the magnetic neutral points, which characterize the magnetic topology of the field lines, can be simply acertained by finding the roots of a polynomial function. The function for the magnetic field is the derivative of a complex potential, the real part of which is a flux function and the imaginary part a magnetic potential of the magnetic field. The inverse function of the complex potential describes the position variable in terms of the complex potential. Field lines are described by a parametric equation obtainable from the afore-mentioned inverse function. Thus, no integration of the differential equation for field lines is needed.

(Keywords: Geomagnetic field, Interplanetary magnetic field, Chapman-Ferraro magnetopause current, Magnetotail current)

1. INTRODUCTION

The earth's magnetosphere is sustained by continual interaction between the earth's magnetic field and the solar wind. The interaction involves induction of magnetopause current and reconnection of geomagnetic and interplanetary field lines. These two features are accounted for in a recent modeling by Yeh (1997). The illustrated field-line configuration is three-dimensional. It accounts for the Chapman-Ferraro magnetopause current by an image dipole placed beyond the dayside magnetopause (cf., Chapman, 1963). And it accounts for the field-line reconnection by a simple superposition of the two magnetic fields (Dungey, 1963).

The Chapman-Ferraro magnetopause current is caused by the solar wind's frontal impingement (Chapman and Ferraro, 1931). In addition to the Chapman-Ferraro current the in-

[1]Institute of Space Science, National Central University, Chung-Li, Taiwan, ROC

[*]*Corresponding author address*: Prof. Tyan Yeh, Institute of Space Science, National Central University, Chung-Li, 320 Taiwan, ROC; E-mail: tyeh@jupiter.ss.ncu.edu.tw

teraction between the solar wind and the earth also induces a magnetotail current, which is caused by lateral pinching of the bypassing flow of the solar wind. It is desirable to include the magnetotail current in the modeling of the magnetosphere. Inclusion of the magnetotail current in a three-dimensional configuration is rather formidable. Instead, we may consider a two-dimensional configuration as a preliminary. The planar magnetic field in the two-dimensional configuration amounts to the noon-midnight profile of the three-dimensional magnetosphere.

The mathematics for planar magnetic fields in current-free regions is facilitated by the usage of complex variables. The two-component magnetic field is to be represented by the derivative of a complex potential which is a function of a complex variable that represents points of the plane. The complex potential is made up of a flux function as its real part and a magnetic potential as its imaginary part. Such usages of analytic functions of a complex variable were fruitfully explored by Dungey (1961) and Hurley (1961) in obtaining the magnetopause profile of a closed magnetosphere. However, their adoption of Ferraro (1960)'s approximation of specular reflection for the incident corpuscular particles as the pressure balance at the magnetopause led to the spurious elongation of the closed magnetosphere to infinity (see Yeh, 1999). Such a spurious geometric openness also appear in Unti and Atkinson (1968)'s inclusion of a neutral sheet current in a two-dimensional closed magnetosphere.

In this paper we study the field-line configuration on the noon-midnight plane in a partially open magnetosphere that includes the magnetotail current. This magnetospheric magnetic field is assumed to be due to four current sources. The part of magnetic field that is due to distant heliospheric currents is accounted for by a uniform magnetic field in southward direction. The part due to the earth's core current is accounted for by a magnetic dipole of southward moment. The part due to the magnetopause current is accounted for by an image dipole of southward moment, which is greater than the earth's dipole moment. The part due to the magnetotail current is accounted for by a line current in duskward direction for its cross-tail current and an image line current in opposite direction for its return current. Each of these four partial magnetic fields has a known complex potential. The resultant complex potential as an analytic function of the complex variable for the position points has a multi-branch inverse function. Field lines are described by a parametric equation obtainable from the afore-mentioned inverse function. Thus, no integration of the differential equation for field lines is needed. The crux of the problem is to ascertain the magnetic neutral points that characterize the magnetic topology of the field lines. In our consideration the function that represents the magnetospheric magnetic field is a rational function. Thus, the magnetic neutral points are simply given by the roots of a polynomial function .

2. PLANAR MAGNETIC FIELD

Generally speaking, a planar magnetic field $\mathbf{1}_x B_x + \mathbf{1}_y B_y$ in a plane (x, y) can be described by a flux function $\Psi(x, y)$. The relationship

$$B_x = \frac{\partial \Psi}{\partial y}, \quad B_y = -\frac{\partial \Psi}{\partial x} \tag{1}$$

assures the satisfaction of the magnetic solenoidality, viz., $\partial B_x/\partial x+\partial B_y/\partial y=0$. On the other hand, in a current-free region the magnetic field can also be described by a magnetic potential $\Omega(x, y)$. The relationship

$$B_x = -\frac{\partial\Omega}{\partial x},\ B_y = -\frac{\partial\Omega}{\partial y} \tag{2}$$

assures the satisfaction of the current-free condition, viz., $\partial B_y/\partial x-\partial B_x/\partial y=0$. It follows from equations (1) and (2) that Ψ and Ω as functions of x and y satisfy Cauchy-Riemann equations $\partial\Psi/\partial x=\partial\Omega/\partial y$, $\partial\Psi/\partial y=-\partial\Omega/\partial x$. Hence, Ψ and Ω satisfy Laplace equation $(\partial^2/\partial x^2+\partial^2/\partial y^2)\Psi=0$, $(\partial^2/\partial x^2+\partial^2/\partial y^2)\Omega=0$. Accordingly, the planar magnetic field can be represented by a complex function $B(z)\equiv B_y(x, y)+iB_x(x, y)$ of the complex variable $z\equiv x+iy$. Indeed, a current-free planar magnetic field can be written $B=-d\Phi/dz$ in terms of a complex potential $\Phi(z)\equiv\Psi(x, y)+i\Omega(x, y)$. Furthermore, $\Phi(z)$ is an analytic function of z. So is its negative-signed derivative $B(z)$.

Since the inverse function of an analytic function of a complex variable is analytic too, the function $z(\Phi)$ inverse to $\Phi(z)$ is an analytic function of Φ. The inverse function $z(\Phi)$ can be multi-valued. Discretion must be exercised to choose the proper branch of the multi-branched function $z(\Phi)$ to obtain the field lines or equipotential lines. The branch points occur at where the derivative $d\Phi/dz$ becomes zero. Thus, the values of Φ at magnetic null points are necessarily branch points of the function $z(\Phi)$. Suppose $f(z, \Phi)$ is a known function that serves to define z in terms of Φ implicitly by the equation $f(z, \Phi)=0$. Then, in addition to satisfying the equation $f(z, \Phi)=0$ the branch-point values of Φ must also satisfy the equation $\partial f/\partial z=0$ in view that both $\partial f/\partial z+(\partial f/\partial\Phi)(d\Phi/dz)$ and $d\Phi/dz$ vanish there. When the function $f(z, \Phi)$ is a polynomial in the first argument z, the branch-point values of Φ as determined by the simultaneous equations $f(z, \Phi)=0$ and $\partial f/\partial z=0$ will render the discriminator of the polynomial vanishing.

In this formulation on analytic functions of complex variable, a uniform magnetic field $B_0(\mathbf{1}_x\cos\theta_0+\mathbf{1}_y\sin\theta_0)$ is represented by a constant function $B_0\exp i(\frac{\pi}{2}-\theta_0)$ with a complex potential given by a linear function $-[B_0\exp i(\frac{\pi}{2}-\theta_0)]\,z$. A magnetic field due to a line current $\mathbf{1}_x\times\mathbf{1}_y I_0$ located at the position z_0 is represented by a rational function $\frac{1}{2\pi}\mu_0 I_0/(z-z_0)$ with a complex potential given by a logarithmic function $-\frac{1}{2\pi}\mu_0 I_0\log(z-z_0)$. And a magnetic field due to a lineal magnetic moment $M_0(\mathbf{1}_x\cos\theta_0+\mathbf{1}_y\sin\theta_0)$ at the position z_0 is represented by $M_0\exp i(\frac{\pi}{2}+\theta_0)/(z-z_0)^2$ with a complex potential given by $M_0\exp i(\frac{\pi}{2}+\theta_0)/(z-z_0)$. Here μ_0 stands for the magnetic permeability.

3. MAGNETOSPHERIC MAGNETIC FIELD IN NOON-MIDNIGHT PLANE

To study the magnetospheric magnetic field in the noon-midnight meridional plane, we choose the origin of the rectangular coordinates (x, y) to be at the earth's center, the x-axis sunward and the y-axis northward. A partially open magnetosphere in two-dimensional plane is represented by the magnetic field

$$B(z) = \frac{M_O}{z^2} - B_I + \frac{M_C}{(z-x_C)^2} - \frac{\mu_0 I_T}{2\pi}\left(\frac{1}{z-x_T} - \frac{1}{z-x_{T'}}\right). \tag{3}$$

The first term M_O/z^2 for a magnetic field due to a southward magnetic moment $-\mathbf{1}_y M_O$ at the origin z=0 accounts for the earth's magnetic field. The second term $-B_I$ for a southward magnetic field $-\mathbf{1}_y B_I$ accounts for the interplanetary magnetic field. The third term $M_C/(z-z_C)^2$ for a magnetic field due to an image dipole of southward magnetic moment $-\mathbf{1}_y M_C$, with $M_C>M_O$, placed at $z=x_C>0$ on the sunward side of the earth accounts for the Chapman-Ferraro magnetopause current. The fourth term for a magnetic field due to a duskward current $-\mathbf{1}_x\times\mathbf{1}_y I_T$ at $z=x_T<0$ on the tailward side of the earth and the image of its return current $\mathbf{1}_x\times\mathbf{1}_y I_T$ placed at $z=x_{T'}>0$ on the sunward side accounts for the magnetotail current. The choice of $x_{T'}\rightarrow+\infty$ will amount to ignoring the return part of the magnetotail current.

The null points of this magnetospheric magnetic field are located at the positions that satisfy the sixth-degree algebraic equation

$$\left[B_I(z-x_T)(z-x_{T'}) - \frac{\mu_0 I_T}{2\pi}(x_{T'}-x_T)\right]z^2(z-x_C)^2 - \left[M_O(z-x_C)^2+M_C z^2\right](z-x_T)(z-x_{T'}) = 0. \tag{4}$$

Two of the six roots have positive real parts. Their imaginary parts will be non-zero when neither B_I nor I_T is too large. The remaining four roots are two negative and two positive. The pair of complex-conjugate roots represent north/south neutral points on the dayside magnetopause. The two negative roots represent x-axis crossing points of two equatorial neutral lines in the tail region. The two positive roots are physically meaningless because they are null points beyond the subsolar point. In fact, the method of images restricts the validity of equation (3) to a domain not beyond the magnetopause. These features follow from the reasoning explained below.

Equation (4) has real-valued coefficients. Its complex-valued roots must be in conjugate pair. From the relationships between coefficients and roots the six roots of equation (4) have a positive sum of $2x_C+x_T+x_{T'}$ and a positive product of $M_O x_C^2(-x_T)x_{T'}/B_I$. The positive sum means that at least one of the six roots has a posiive real part, and accordingly the positive product means that at least two of the six roots have positive real parts. The two roots that have positive real parts do have non-zero imaginary parts when both B_I and I_T are zero. More details are revealed in the following two limiting cases. In the limit case of $B_I\rightarrow 0$, two of the six roots become $\pm\infty$ and hence the remaining four roots become $x_C(M_O\pm i\sqrt{M_O M_C})/(M_O+M_C)$, x_T, $x_{T'}$ when I_T is zero. In the limit case of $M_C\rightarrow M_O$ and $x_T+x_{T'}\rightarrow x_C$, the sixth-degree equation becomes a cubic equation

$$\left\{B_I\left[(z-\tfrac{1}{2}x_C)^2 - \tfrac{1}{4}x_C^2\right]^2 - 2M_O\left[(z-\tfrac{1}{2}x_C)^2 + \tfrac{1}{4}x_C^2\right]\right\}\left[(z-\tfrac{1}{2}x_C)^2 - (\tfrac{1}{2}x_C - x_T)^2\right]$$

$$-\frac{\mu_0 I_T}{2\pi}(x_C - 2x_T)\ [(z-\tfrac{1}{2}x_C)^2 - \tfrac{1}{4}x_C^2]^2 = 0 \tag{5}$$

in $(z-\frac{1}{2}x_C)^2$. The three roots for $(z-\frac{1}{2}x_C)^2$ have a positive sum of $\frac{1}{2}x_C^2+(\frac{1}{2}x_C-x_T)^2+(x_C-2x_T)\mu_0 I_T/2\pi B_I+2M_O/B_I$ and a product of $\frac{1}{4}x_C^2(\frac{1}{2}x_C-x_T)[\frac{1}{4}x_C^2(\frac{1}{2}x_C-x_T+\mu_0 I_T/\pi B_I)-(x_C-2x_T)M_O/B_I]$. When B_I is not too large, the sum is greater than x_C^2 and the product is negative. Hence one of the three roots for $(z-\frac{1}{2}x_C)^2$ is negative and two are positive. The negative root for $(z-\frac{1}{2}x_C)^2$, varying from $-\frac{1}{4}x_C^2$ to 0 when B_I and I_T increase from 0 to larger values that make $B_I+\mu_0 I_T/\pi(\frac{1}{2}x_C-x_T)$ equal to $8M_O/x_C^2$, yields a pair of complex-conjugate roots for z. One of the two positive roots for $(z-\frac{1}{2}x_C)^2$, varying from $+\infty$ to $\frac{1}{4}x_C^2$ when B_I and I_T increase from 0 to $+\infty$, yields a negative and a positive roots for z. So does the other positive root for $(z-\frac{1}{2}x_C)^2$, which varies from $(\frac{1}{2}x_C-x_T)^2$ to $\frac{1}{4}x_C^2$. The pair of complex-conjugate roots for z represent the north/south neutral points. They move from $\frac{1}{2}x_C \pm i\frac{1}{2}x_C$ to coalesce at $\frac{1}{2}x_C+i0$. The former negative root for z represents the farther crossing point. It moves from $-\infty$ toward the earth's dipole. The latter negative root for z represents the near-earth crossing point. It moves from x_T toward the dipole.

The magnetospheric magnetic field given by equation (3) has the complex potential

$$\Phi(z) = \frac{M_O}{z} + B_I z + \frac{M_C}{z - x_C} + \frac{\mu_0 I_T}{2\pi}\log\frac{z - x_T}{z - x_{T'}}. \tag{6}$$

Equation (6) defines the inverse function $z(\Phi)$ implicitly. The function $z(\Phi)$ is multi-valued, with branch points being associated with the magnetic null points where the derivative $d\Phi/dz$ vanishes. The multi-branch function $z(\Phi)$ can be obtained explicitly when the logarithmic term is absent. In the latter case z can be obtained in terms of Φ by solving a cubic equation with complex-valued coefficients.

4. FIELD-LINE CONFIGURATION WITHOUT MAGNETOTAIL CURRENT

It is instructive to examine field-line configuration without magnetotail current before we deal with field-line configuration with magnetotail current.

First, in the absence of the magnetotail current as well as the interplanetary magnetic field equation (3) reduces to

$$B(z) = \frac{M_O}{z^2} + \frac{M_C}{(z - x_C)^2}. \tag{7}$$

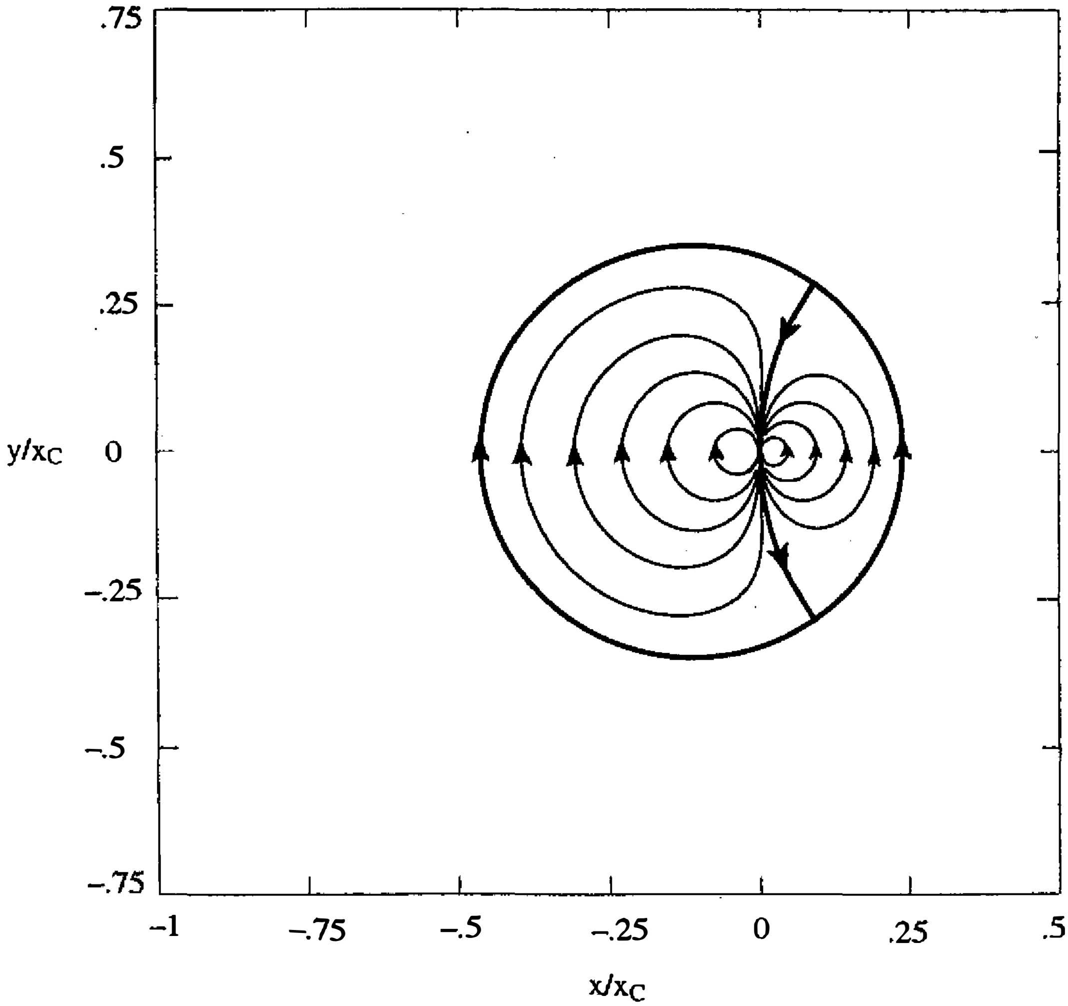

Fig.1. A closed magnetosphere without magnetotail current. $M_c/M_o=10$. Subsolar and antisolar traces of magnetopause as well as cusp-to-dipole field lines are indicated by thick lines.

It describes a closed magnetosphere. From the quadratic equation

$$(M_O + M_C)z^2 - 2M_O x_C z + M_O x_C^2 = 0 \tag{8}$$

we find two null points at $z=x_0 \pm iy_0$, with

$$x_0 = \frac{M_O}{M_O+M_C}x_C, \quad y_0 = \frac{\sqrt{M_O M_C}}{M_O+M_C}x_C.$$

They are the north/south neutral points on the dayside magnetopause of the closed magnetosphere.

The reduced complex potential

$$\Phi(z) = \frac{M_O}{z} + \frac{M_C}{z - x_C} \tag{9}$$

has the values $\Psi_0 \pm i\Omega_0$ with

$$\Psi_0 = -\frac{M_C - M_O}{x_C}, \quad \Omega_0 = -2\frac{\sqrt{M_O M_C}}{x_C},$$

at the neutral points $x_0 \pm iy_0$. The inverse function $z(\Phi)$ can be obtained explicitly from the quadratic equation

$$\Phi z^2 - (M_O + M_C + x_C\Phi)z + M_O x_C = 0\,. \tag{10}$$

The two-branch function

$$z(\Phi) = \frac{1}{2}x_C + \frac{1}{2}\frac{M_O + M_C}{\Phi} + \frac{1}{2}\left[x_C^2 + 2\frac{M_C - M_O}{\Phi}x_C + \frac{(M_O + M_C)^2}{\Phi^2}\right]^{1/2}, \tag{11}$$

has branch points at $\Phi = \Psi_0 \pm i\Omega_0$. There the quadratic discriminator, which is a quadratic in Φ, vanishes. The two-branch parametric function $z|_{\Phi=\Psi_0+i\Omega}$ with Ω varying from positive $-\Omega_0$ to negative Ω_0 describes the separatrix field lines, which constitute the noon-midnight meridional trace of the entire magnetopause of the closed magnetosphere. The subsolar separatrix field line is given by one branch of the parametric function and the antisolar separatrix field line is given by another branch, both from the south neutral point to the north neutral point. The subsolar/antisolar points $x_C/(1 \pm \sqrt{M_C/M_O}) + i0$ are given by the two values of $z|_{\Phi=\Psi_0+i0}$. with Ω equal to 0. The two cusp-to-earth field lines are given by one of the two branches of the parametric function $z|_{\Phi=\Psi_0+i\Omega}$. The north cusp-to-earth field line has negative Ω varying from Ω_0 to $-\infty$ and the south cusp-to-earth field line has positive Ω varying from $-\Omega_0$ to $+\infty$ (see Yeh, 1999). Figure 1 shows the field lines for the obtained closed magnetosphere with $M_C/M_O = 10$.

Next, in the presence of the interplanetary magnetic field, without the magnetotail current, equation (3) reduces to

$$B(z) = \frac{M_O}{z^2} - B_I + \frac{M_C}{(z - x_C)^2}\,. \tag{12}$$

It describes a partially open magnetosphere. The null points of this magnetic field are located at the positions that satisfy the quartic equation

$$B_I z^4 - 2B_I x_C z^3 + (B_I x_C^2 - M_O - M_C)z^2 + 2M_O x_C z - M_O x_C^2 = 0. \tag{13}$$

It can be factored, by the classical Ferrari's method, into two quadratic equations

$$z^2-(x_C-\sqrt{\tau})z+\frac{M_C-M_O}{2B_I}\frac{x_C}{\sqrt{\tau}}-\frac{M_O+M_C}{2B_I}-\frac{1}{2}x_C\sqrt{\tau}+\frac{1}{2}\tau=0 \tag{14}$$

and

$$z^2-(x_C+\sqrt{\tau})z-\frac{M_C-M_O}{2B_I}\frac{x_C}{\sqrt{\tau}}-\frac{M_O+M_C}{2B_I}+\frac{1}{2}x_C\sqrt{\tau}+\frac{1}{2}\tau=0\,, \tag{15}$$

in which τ is a positive intermediary that satisfies the cubic equation

$$\tau^3-\left(x_C^2+2\frac{M_O+M_C}{B_I}\right)\tau^2+\left(2x_C^2+\frac{M_O+M_C}{B_I}\right)\frac{M_O+M_C}{B_I}\tau-\left(\frac{M_C-M_O}{B_I}\right)^2x_C^2=0\,. \tag{16}$$

This cubic equation for τ has a positive discriminator when B_I is less than $(M_O^{1/3}+M_C^{1/3})^3/x_C^2$. In the range $0<B_I<(M_O^{1/3}+M_C^{1/3})^3/x_C^2$ of our interest, equation (14) has two complex-conjugate roots $z=x_0\pm iy_0$ with

$$x_0=\frac{1}{2}x_C-\frac{1}{2}\sqrt{\tau},\quad y_0=\frac{1}{2}\sqrt{2\frac{M_C-M_O}{B_I}\frac{x_C}{\sqrt{\tau}}-2\frac{M_O+M_C}{B_I}-x_C^2+\tau}$$

and equation (15) has a negative root $z=x_{0'}$ with

$$x_{0'}=\frac{1}{2}x_C+\frac{1}{2}\sqrt{\tau}-\frac{1}{2}\sqrt{2\frac{M_C-M_O}{B_I}\frac{x_C}{\sqrt{\tau}}+2\frac{M_O+M_C}{B_I}+x_C^2-\tau}\,.$$

The complex-conjugate roots represent the north/south neutral points on the sunward magnetopause and the negative root represents the x-axis crossing point of an equatorial neutral line in the tail region. The other root of equation (15), being greater than x_C, has no physical meaning because it is beyond the image dipole on the sunward side of the subsolar point. In the limit of $B_I\to0$, we have $\sqrt{\tau}\to x_C(M_C-M_O)/(M_O+M_C)$, hence $x_0\to x_CM_O/(M_O+M_C)$, $y_0\to x_C\sqrt{M_OM_C}/(M_O+M_C)$ and $x_{0'}\to-\infty$. In the other limit of $B_I\to(M_O^{1/3}+M_C^{1/3})^3/x_C^2$, we have $\sqrt{\tau}\to x_C(M_C^{1/3}-M_O^{1/3})/(M_O^{1/3}+M_C^{1/3})$, hence $x_0\to x_CM_O^{1/3}/(M_O^{1/3}+M_C^{1/3})$, $y_0\to0$ and $x_{0'}\to-x_CM_O^{1/3}/(\sqrt{M_O^{2/3}+M_O^{1/3}M_C^{1/3}+M_C^{2/3}}+M_C^{1/3})$. Thus the north/south neutral points move from the locations where the neutral points of the closed magnetosphere reside to coalesce at the x-axis and the crossing point moves from the antisolar infinity toward the earth's dipole. Finally, in the limit of $M_C\to M_O$, we have $\tau\to x_C^2(M_C-M_O)^2/4(x_C^2B_I+M_O)M_O$, hence $x_0\to\frac{1}{2}x_C$, $y_0\to\sqrt{\sqrt{(x_C^2+M_O/B_I)M_O/B_I}-\frac{1}{4}x_C^2-M_O/B_I}$ $x_{0'}\to\frac{1}{2}x_C-\sqrt{\sqrt{(x_C^2+M_O/B_I)M_O/B_I}+\frac{1}{4}x_C^2+M_O/B_I}$. These magnetic null points can be obtained directly from equation (5) which simplifies to a quadratic equation and a linear equation in $(z-\frac{1}{2}x_C)^2$ explicitly when I_T is zero. In this symme-

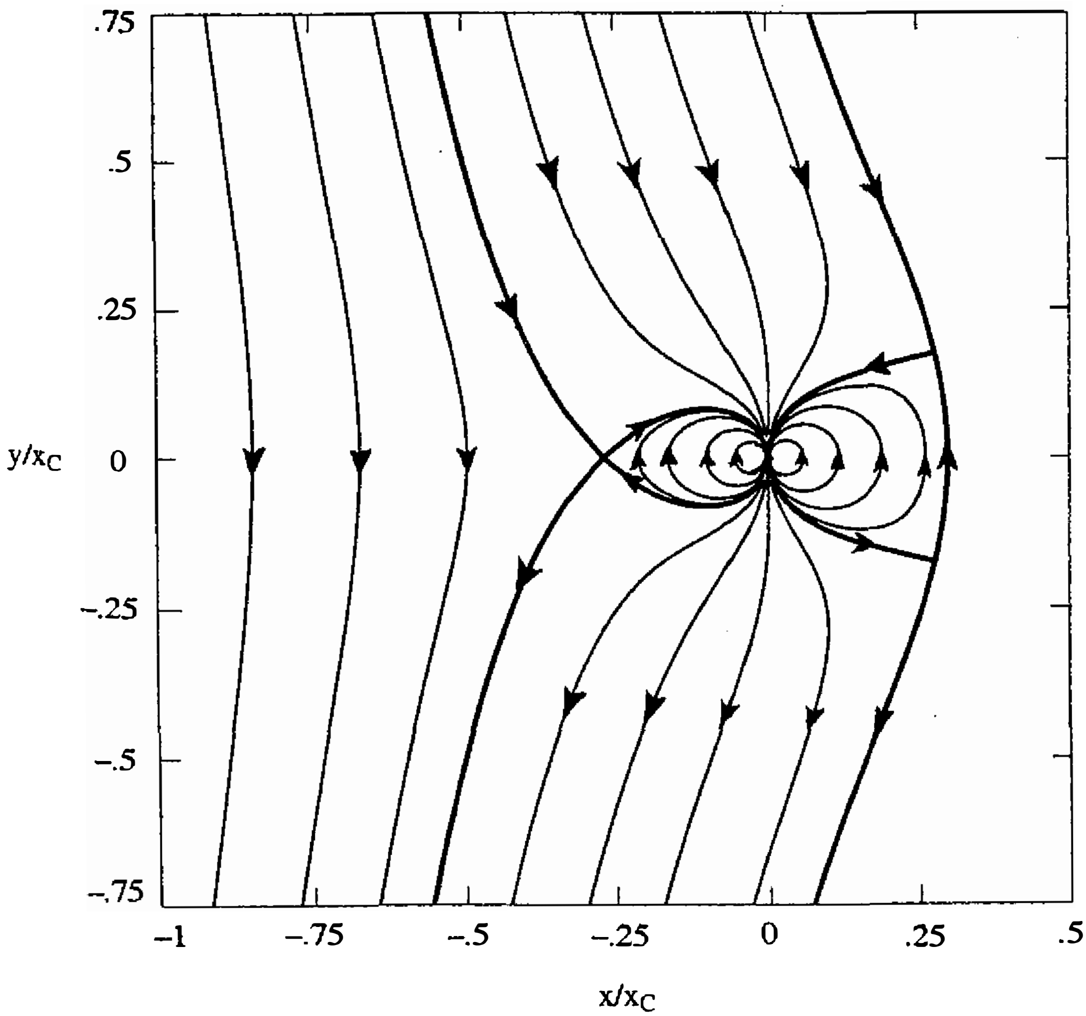

Fig.2. A partially open magnetosphere without magnetotail current. $M_C/M_0=10$ and $B_I x_C^2/M_0=20$. Traces of separatrix surfaces are indicated by thick lines.

try case the north/south neutral points are on the midplane $x=\frac{1}{2}x_C$ between the earth' dipole and the image dipole.

The reduced complex potential

$$\Phi(z)=\frac{M_O}{z}+B_I z+\frac{M_C}{z-x_C} \tag{17}$$

has the values $\Psi_0 \pm i\Omega_0$ with

$$\Psi_0 = M_O \frac{x_0}{x_0^2+y_0^2} + M_C \frac{x_0 - x_C}{(x_0-x_C)^2+y_0^2} + B_I x_0,$$

$$\Omega_0 = M_O \frac{-y_0}{x_0^2+y_0^2} + M_C \frac{-y_0}{(x_0-x_C)^2+y_0^2} + B_I y_0$$

at the north/south neutral points $x_0 \pm iy_0$ and the value $\Psi_{0'}+i\Omega_{0'}$ with

$$\Psi_{0'} = \frac{M_O}{x_{0'}} + \frac{M_C}{x_{0'} - x_C} + B_I x_{0'}, \quad \Omega_{0'} = 0$$

at the crossing point $x_{0'}+i0$. The inverse function $z(\Phi)$ can be obtained explicitly from the cubic equation

$$B_I z^3 - (B_I x_C + \Phi)z^2 + (M_O + M_C + x_C\Phi)z - M_O x_C = 0 \tag{18}$$

by the classical Cardano's method. It is

$$z(\Phi) = \frac{1}{3}\left(x_C + \frac{\Phi}{B_I}\right) + \left[-\frac{1}{2}Q + \left(\frac{1}{4}Q^2 + \frac{1}{27}P^3\right)^{1/2}\right]^{1/3} - \frac{\frac{1}{3}P}{\left[-\frac{1}{2}Q + \left(\frac{1}{4}Q^2 + \frac{1}{27}P^3\right)^{1/2}\right]^{1/3}}, \tag{19}$$

in which

$$P(\Phi) = \frac{M_O+M_C}{B_I} + x_C\frac{\Phi}{B_I} - \frac{1}{3}\left(x_C + \frac{\Phi}{B_I}\right)^2,$$

$$Q(\Phi) = -x_C\frac{M_O}{B_I} + \frac{1}{3}\left(x_C + \frac{\Phi}{B_I}\right)\left(\frac{M_O+M_C}{B_I} + x_C\frac{\Phi}{B_I}\right) - \frac{2}{27}\left(x_C + \frac{\Phi}{B_I}\right)^3.$$

The function $z(\Phi)$ has four branch points, at which the cubic discriminator $\frac{1}{4}Q^2+\frac{1}{27}P^3$ (which is only a quartic in Φ) vanishes. The branch points associated with the north/south neutral points have $\Phi=\Psi_0 \pm i\Omega_0$ and the branch point associated with the crossing point has $\Phi=\Psi_{0'}+i0$.

The three-branch parametric function $z|_{\Phi=\Psi_0+i\Omega}$ describes the field lines that emanate from or terminate at the north/south neutral points. The subsolar separatrix field line, from the south neutral point to the north neutral point, has Ω varying from positive $-\Omega_0$ to negative Ω_0. The subsolar point is given by one of the three values of $z|_{\Phi=\Psi_0+i0}$ with Ω equal to zero. The north antisolar separatrix field line has Ω varying from negative Ω_0 to positive infinity. The antisolar point at north infinity is given by one of the three values of $z|_{\Phi=\Psi_0+i\infty}$ with Ω equal to $+\infty$. The south antisolar separatrix field line has Ω varying from positive $-\Omega_0$ to negative infinity. The antisolar point at south infinity is given by one of the three values of $z|_{\Phi=\Psi_0+i\infty}$ with Ω equal to $-\infty$. The two cusp-to-earth field lines are given by another branch of the parametric function $z|_{\Phi=\Psi_0+i\Omega}$. The north cusp-to-earth field line has Ω varying from negative

Ω_0 to $-\infty$ and the south cusp-to-earth field line has Ω varying from positive $-\Omega_0$ to $+\infty$. Likewise, the parametric function $z\big|_{\Phi=\Psi_{0'}+i\Omega}$ describes the four field lines that emanate from or terminate at the crossing point. Two of them have negative Ω varying from 0 to $-\infty$, the other two have positive Ω varying from 0 to $+\infty$. Figure 2 shows the field lines for the obtained patially open magnetosphere with $M_C/M_0=10$ and $B_I x_C^2/M_0=20$.

5. A CLOSED MAGNETOSPHERE WITH MAGNETOTAIL CURRENT

Now we include the magnetotail current. In the absence of the interplanetary magnetic field equation (3) reduces to

$$B(z) = \frac{M_O}{z^2} + \frac{M_C}{(z-x_C)^2} - \frac{\mu_0 I_T}{2\pi}\left(\frac{1}{z-x_T} - \frac{1}{z-x_{T'}}\right). \qquad (20)$$

It describes a closed magnetosphere with magnetotail current. The null points of this magnetic field for a closed magnetosphere can be found from the quartic equation

$$\begin{aligned}&[M_O+M_C+\frac{\mu_0 I_T}{2\pi}(x_T'-x_T)]z^4 - \left\{2[M_O+\frac{\mu_0 I_T}{2\pi}(x_T'-x_T)]x_C + (M_O+M_C)(x_T+x_T')\right\}z^3\\ &\quad+\left\{[M_O+\frac{\mu_0 I_T}{2\pi}(x_T'-x_T)]x_C^2 + 2M_O x_C(x_T+x_T') + (M_O+M_C)x_T x_T'\right\}z^2\\ &\quad - M_O x_C[x_C(x_T+x_T') + 2x_T x_T']z + M_O x_C^2 x_T x_T' = 0. \qquad (21)\end{aligned}$$

Two of the four roots have positive real parts. They will have non-zero imaginary parts if I_T is in the range between 0 and a certain value that makes the discriminator of the associated cubic equation vanishing, in exact analogy with the quartic equation (13). The remaining two roots are one negative and one positive. The pair of complex-conjugate roots $x_0\pm iy_0$ represent the north/south neutral points on the dayside magnetopause of the closed magnetosphere. The negative root $x_{0''}$, with a value between x_T and 0, represents the crossing point of a near-earth neutral line between the cross-tail current line and the earth's dipole. The positive root, with a value between $x_{T'}$ and x_C, has no physical meaning. In the limit of $I_T\to 0$, the four roots are $x_C(M_0\pm i\sqrt{M_O M_C})/(M_0+M_C)$, x_T and $x_{T'}$. More informative is the symmetry case of $M_C=M_O$ and $x_T+x_{T'}=x_C$. In that case equation (21) reduces to a quadratic equation in $(z-\frac{1}{2}x_C)^2$, which is nothing but equation (5) with B_I set to 0. The root for $(z-\frac{1}{2}x_C)^2$ that increases from $-\frac{1}{4}x_C^2$ to 0 when $\frac{1}{2\pi}\mu_0 I_T$ increases from 0 to $2M_0(x_C-2x_T)/x_C^2$ yields the pair of complex-conjugate roots $x_0\pm iy_0$ for z. These north/south neutral points move from $\frac{1}{2}x_C\pm i\frac{1}{2}x_C$ to coalesce at $\frac{1}{2}x_C+i0$. The other roots for $(z-\frac{1}{2}x_C)^2$ that decreases from $(\frac{1}{2}x_C-x_T)^2$ to 0 when I_T increases from 0 to $+\infty$ yields the negative root $x_{0''}$ for z. This near-earth crossing point moves from x_T toward the

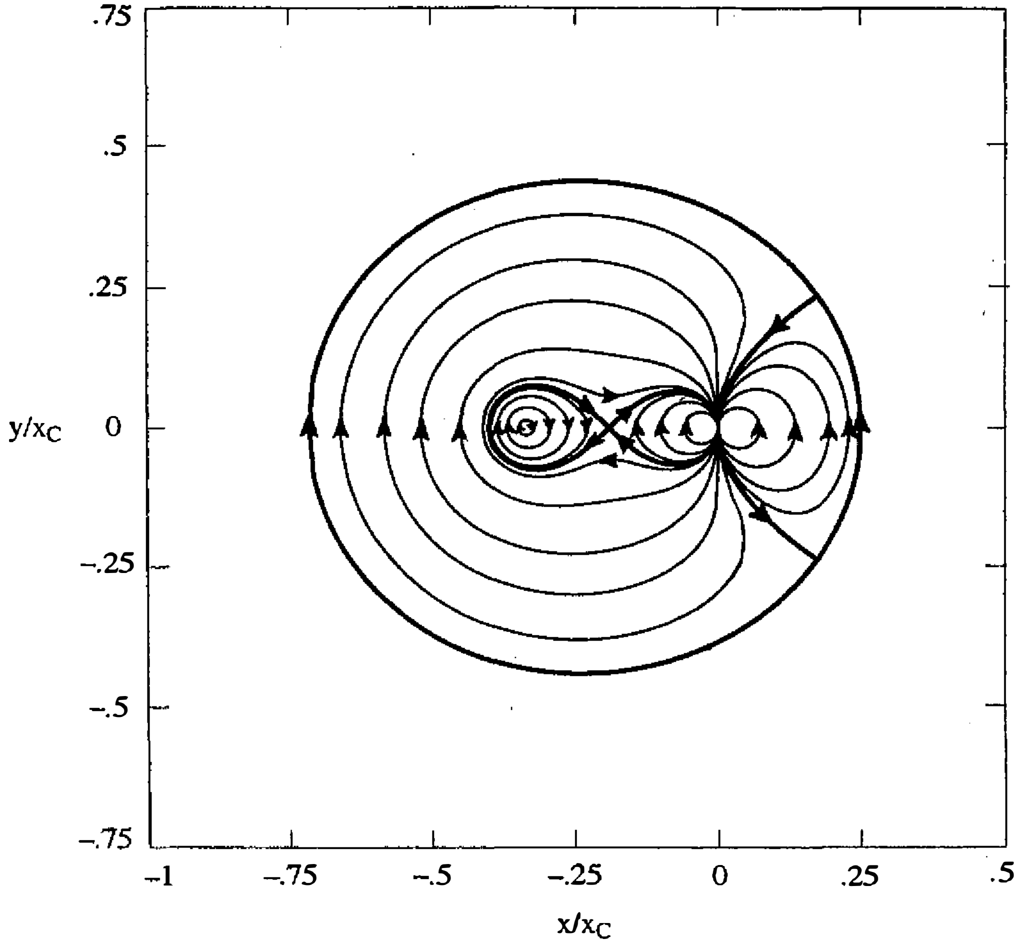

Fig.3. A closed magnetosphere with magnetotail current. M_c/M_0=10 and $(\mu_0 I_T/2\pi)x_c/M_0$=5, x_T/x_c=–0.33. Subsolar and antisolar traces of magnetopause as well as cusp-to-dipole field lines and other traces of separatrix surfaces are indicated by thick lines.

earth's dipole.

The complex potential

$$\Phi(z)=\frac{M_O}{z}+\frac{M_C}{z-x_C}+\frac{\mu_0 I_T}{2\pi}\log\frac{z-x_T}{z-x_{T'}} \qquad (22)$$

has branch points at the north/south neutral points and the crossing point of the equatorial neutral line. On the x-axis the flux function varies as $M_0/x+M_c/(x-x_c)+\frac{1}{2\pi}\mu_0 I_T \log(|x-x_T|/|x-x_{T'}|)$. It attains a maximal value at the crossing point and becomes $-\infty$ at the cross-tail current line. The magnetic potential attains the value of 0 on the segment $-\infty<x<x_T$ and the

discontinuous value $\mu_0 I_T y/|y|$ on the segment $x_T<x<x_{T'}$.

Figure 3 shows the field lines for the obtained closed magnetosphere with $M_C/M_0=10$ and $(\mu_0 I_T/2\pi)x_C/M_0=5$, $x_T/x_C=-0.33$, $x_{T'}=+\infty$. There is a magnetic island of field lines that encircle the magnetotail current. These isolated field lines do not link through the earth.

6. A PARTIALLY OPEN MAGNETOSPHERE WITH MAGNETOTAIL CURRENT

The most general case of our ultimate interest is a partially open magnetosphere with magnetotail current in the presence of the magnetotail current as well as the interplanetary magnetic field.

Without loss of generality we shall ignor the image current (viz., $x_{T'}\to+\infty$). The ignoring makes one of the physically meaningless roots of equation (4) recede to $+\infty$. The simplified magnetic field

$$B(z)=\frac{M_O}{z^2}-B_I+\frac{M_C}{(z-x_C)^2}-\frac{\mu_0 I_T}{2\pi}\frac{1}{z-x_T} \tag{23}$$

has five null points. They are located at the positions that satisfy the fifth-degree equation

$$\left[B_I(z-x_T)+\frac{\mu_0 I_T}{2\pi}\right]z^2(z-x_C)^2-\left[M_O(z-x_C)^2+M_C z^2\right](z-x_T)=0. \tag{24}$$

The five roots have a positive product $M_0 x_C^2(-x_T)/B_I$. Hence one of the roots is positive, being greater than x_C. Once this physically meaningless positive root (or alternatively one of the negative roots for the crossing points of neutral lines) is numerically found, equation (24) can be factored to yield a quartic equation. The resulting quartic equation, which can be solved by Ferrari's quartic formula, has two complex-conjugate roots and two negative roots if neither B_I nor I_T is too large. The complex-conjugate roots $x_0\pm iy_0$ represent the north/south neutral points on the sunward magnetopause of the partially open magnetosphere. The two negative roots represent two crossing points of neutral lines in the tail region. The farther crossing point $x_{0'}+i0$ is on the tailward side of the line current whereas the near-earth crossing point $x_{0''}+i0$ is between the line current and the earth's dipole.

The simplified complex potential

$$\Phi(z)=\frac{M_O}{z}+B_I z+\frac{M_C}{z-x_C}+\frac{\mu_0 I_T}{2\pi}\log(z-x_T). \tag{25}$$

has branch points at the north/south neutral points and the two crossing points of neutral lines. On the x-axis the flux function varies as $M_0/x+B_I x+M_C/(x-x_C)+\frac{1}{2\pi}\mu_0 I_T \log(|x-x_T|/|x-x_{T'}|)$. It attains maximal values at the two crossing points and becomes $-\infty$ at the cross-tail current line. The two crossing points will be connected by two field lines, that delineate a magnetic island, when the flux function has the same value at them. This requires a certain compatibility constraint

$$\frac{M_O}{x_{O'}} + B_I x_{O'} + \frac{M_C}{x_{O'} - x_C} + \frac{\mu_0 I_T}{2\pi}\log(x_{O'} - x_T) = \frac{M_O}{x_{O''}} + B_I x_{O''} + \frac{M_C}{x_{O''} - x_C} + \frac{\mu_0 I_T}{2\pi}\log(x_{O''} - x_T) \qquad (26)$$

among the characterizing parameters.

Figure 4 shows the field lines for the obtained partially open magnetosphere with $M_C/M_O=10$, $B_I x_C^2/M_O=20$ and $(\mu_0 I_T/2\pi)x_C/M_O=5$, $x_T/x_C=-0.33$ There is a magnetic island of isolated field lines that encircle the magnetotail current.

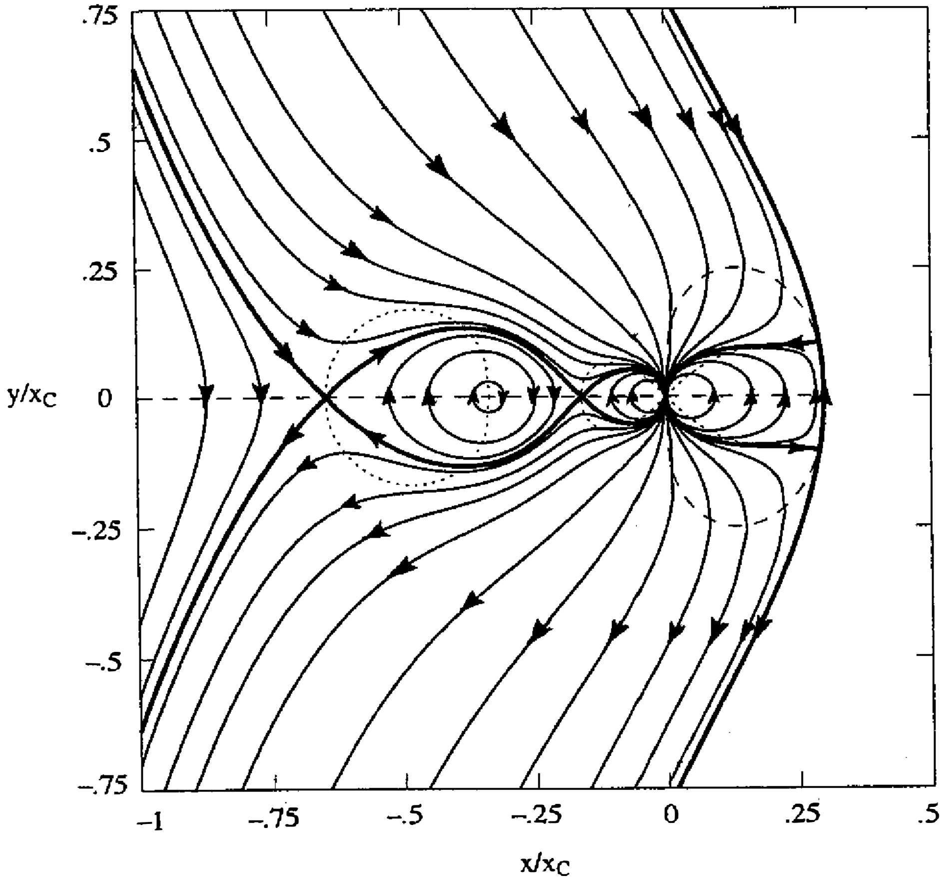

Fig.4. A partially open magnetosphere with magnetotail current. $M_C/M_O=10$, $B_I x_C^2/M_O=20$ and $(\mu_0 I_T/2\pi)x_C/M_O=5$, $x_T/x_C=-0.33$. Traces of separatrix surfaces are indicated by thick lines. Dashed lines indicate where B_x vanishes and dotted lines where B_y vanishes. North/south neutral points and crossing points of two equatorial neutral lines are at intersections of dashed and dotted lines.

8. DISCUSSION

In our study of a two-dimensional magnetosphere, the cross-tail current is represented by a line current. It is encircled by field lines, that do not link through the earth. These isolated field lines constitute a magnetic island. The existence of the magnetic island incurs a near-earth neutral line in the magnetotail. The near-earth neutral line delineates the edge of the plasma sheet in the tail region. The neutral sheet of the magnetotail is nothing but a magnetic island in the form of a thin sheet. When the magnetic island is deformed into a neutral sheet, the cross-tail current must spread with suitable spatial distribution so that the isolated field lines are confined to a thin sheet. In the limit of zero width for the magnetic island, the isolate field lines appear as if they were non-existent.

Finally we remark that in our study we represent the Chapman-Ferraro magnetopause current by a magnetic dipole as an image current . The image dipole has a negative-order moment. So it is located outside the magnetopause on which the actual current resides. Therefore, the obtained magnetic field is valid only in the interior region inside the closed surface formed by the entire magnetopause. The magnetopause in a closed magnetosphere is delineated by the field lines that emanate from the south neutral point and terminated at the north neutral point (see Fig. 1 and Fig. 3). As for a partially open magnetosphere, the front part of the magnetopause is covered by field lines emanated from or terminated at neutral points. The rear part of the magnetopause is an equipotential surface of zero magnetic potential.

Acknowledgments This research was supported by the National Science Council under grant NSC 88-2111-M-008-006-AP8 to the National Central University.

REFERENCES

Chapman, S., 1963: Solar plasma, geomagnetism and aurora. Geophysics–The Earth's Environment. In: C. DeWitt, J. Hieblot, and A. Lebeau (Eds.), Gordon and Breach Science Publishers, New York, 372–502.

Chapman, S., and V. C. A. Ferraro, 1931: A new theory of magnetic storms. *Terr. Mag. Atmosph. Elect.*, **36**, 77–97.

Dungey, J. W., 1961: The steady state of the Chapman-Ferraro problem in two dimensions. *J. Geophys. Res.*, **66**, 1043–1047.

Dungey, J. W., 1963: The structure of the exosphere or the adventures in velocity space. Geophysics–The Earth's Environment. In: C. DeWitt, J. Hieblot, and A. Lebeau (Eds.), Gordon and Breach Science Publishers, New York, 503–550.

Ferraro, V. C. A., 1960: An approximate method of estimating the size and shape of the stationary hollow carved out of a neutral ionized stream of corpuscles impinging on the geomagnetic field. *J. Geophys. Res.*, **65** , 3951–3953.

Hurley, J., 1961: Interaction of a streaming plasma with the magnetic field of a two-dimensional dipole. *Phys. Fluids*, **4**, 854–859.

Unti, T., and G. Atkinson, 1968: Two-dimensional Chapman-Ferraro problem with neutral sheet. *J. Geophys. Res.*, **73**, 7319–7327.

Yeh, T., 1997: Magnetospheric structure with a southward interplanetary magnetic field. *J. Geophys. Res.*, **102**, 51–56.

Yeh, T., 1999: General solution of two-dimensional Chapman-Ferraro problem of magnetospheric cavity. *TAO*, **10**, 397–414.