TAO, Vol. 10, No. 2, 397-414, June 1999

General Solution of Two-Dimensional Chapman-Ferraro Problem of Magnetospheric Cavity

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(Manuscript received 31 July 1998, in final form 16 March 1999)

ABSTRACT

The famous Chapman-Ferraro problem for the determination of the magnetospheric cavity is elucidated for its two-dimensional solutions in mathematical detail. The general solution presented in this paper amounts to a complex variable extension of the well-known Poisson's integral formula for the Dirichlet problem for a half-plane or a circle domain. It provides the needed methodology for obtaining desirable solutions that avoid the defects inherent in the Dungey-Hurley solution, which uses Ferraro's approximation of pressure balance as the boundary condition.

(Key words: Chapman-Ferraro problem, Magnetospheric cavity, Solar wind, Geomagnetic field)

1. INTRODUCTION

The pioneering work of Chapman and Ferraro on geomagnetic storms in 1930's started modern research on the earth's magnetosphere, although the latter name was coined by Gold much later in the 1950s, at the dawn of space age. The 'cavity', as it was called by Chapman and Ferraro (1931), for the void carved out of the streams of solar corpuscles is a magnetospheric region not reached by the solar wind, in today's nomenclature. Before the ultimate acceptance of the suggestion by Dungey (1961a) that field line reconnection between interplanetary and terrestrial magnetic fields takes place to link the earth to interplanetary space magnetically, the cavity was thought to be covered on its entire surface by a sheet-like current to form a closed magnetosphere. This Chapman-Ferraro current on the magnetopause is induced by the impingement of the solar wind. It produces a magnetic field that adds to the earth's dipolar field so that the resultant magnetic field has a magnetic pressure to balance the dynamic pressure exerted by the solar wind.

Instead of determining the distribution of the dynamic pressure of the impinging solar wind from magnetohydrodynamic consideration of the interaction between the solar wind and the earth, Ferraro (1960) adopted the approximation of specular reflection for the incident corpuscular particles. Such a simplification reduces the mathematical determination of the

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magnetospheric cavity to a problem of free boundary on which pressure balance is to be satisfied. This constitutes the famous Chapman-Ferraro problem.

To date, the Chapman-Ferraro problem has been solved only for its two-dimensional version to the extent of fully revealing the solution's dependence on the earth's magnetic moment and the solar wind's mass density and flow velocity. Based on the insights provided by this two-dimensional solution, approximate solutions of the three-dimensional Chapman-Ferraro problem have been attempted, using various numerical schemes for some values of the parameters (e.g., Mead and Beard, 1964). These exemplary solutions have served as the framework for building reference models to organize observational data on the magnetosphere (see Siscoe, 1988). Strictly speaking, solutions for open magnetosphere should be used to build the reference models because the earth's magnetosphere is now regarded as open, with interconnective field lines. But, until now, the corresponding ChapmanFerraro-Dungey problem for a partially open magnetosphere has not yet been formulated. Recently, Yeh (1997) elucidated the magnetic topology of the magnetosphere by a model of partially open magnetosphere, which has a front part much like a closed magnetosphere and a rear part like a fully open magnetosphere. Thus, a re-examination of the Chapman-Ferraro problem is desirable to provide some useful clues for the formulation of the ChapmanFerraro-Dungey problem. For this purpose, we shall present the general solution of the two-dimensional Chapman-Ferraro problem.

The exact solution of the two-dimensional Chapman-Ferraro problem with Ferrao's approximation of pressure balance as the boundary condition was given by Dungey (1961b) and Hurley (1961) in different mathematical forms. They applied different transformations to the Laplace equation that governs the scalar potential and flux function of the magnetic field. They gave the solutions explicitly and showed that the given solutions indeed satisfy the prescribed boundary conditions. However, neither of them revealed how the explicit solutions were obtained.

In this paper, we shall elucidate the methodologies for obtaining the general solution for arbitrary boundary conditions. The general solution amounts to Poisson's integral formula for a Dirichlet problem with a half-plane or a circle domain. Conformal transformations, such as the Schwarz-Christoffel transformation, are utilized in the mathematical process. This general solution enables us to obtain desirable solutions with other boundary conditions.

2. CHAPMAN-FERRARO PROBLEM IN TWO DIMENSIONS

We use rectangular coordinates (x, y) for the noon-midnight meridional plane, with the origin at the earth's center, the x-axis sunward and the y-axis northward. Each coordinate point can be represented by a complex number $z \equiv x+iy$. The magnetic field $\mathbf{1}_x B_x + \mathbf{1}_y B_y$ can be represented by a complex function $B(z) \equiv B_y(x, y)+iB_x(x, y)$. For a current-free magnetic field, we can write $B=-d\Phi/dz$ in terms of a complex potential $\Phi(z) \equiv \Psi(x, y)+i\Omega(x, y)$, with Ψ being a flux function and Ω a scalar potential. The relations

$$B_x = \frac{\partial \Psi}{\partial y} = -\frac{\partial \Omega}{\partial x}, \tag{1}$$

$$B_y = -\frac{\partial \Psi}{\partial x} = -\frac{\partial \Omega}{\partial y} \tag{2}$$

indicate that Ψ and Ω as functions of x and y satisfy Cauchy-Riemann equations. Hence, each of Ψ and Ω satisfies Laplace equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Psi = 0, \tag{3}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\Omega = 0. \tag{4}$$

Accordingly, Φ is an analytic function of the complex variable z. In our usage of complex variables the earth's dipole field has the complex potential M_0/z, with M_0 being the magnitude of the earth's southward dipole moment.

The problem at hand is to find a desirable Φ that behaves asymptotically like a dipole near the origin (viz., $\Phi \rightarrow M_0/z$ as $z \rightarrow 0$) and also satisfies the boundary condition specified on the free boundary for the magnetopause. We may choose Ψ to have a value of 0 on the field lines that contain the north/south neutral points and Ω to have a value of 0 on the equipotential lines that contain the subsolar/antisolar stagnation points. Conceptually, the solution should have field lines and equipotential lines like what is depicted in Figure 1 as far as topology is concerned.

In Ferraro's (1960) approximation of specular reflection for incident particles, the solar wind's dynamic pressure on the magnetopause is taken as $p_I \cos^2\chi$, with $p_I \equiv 2(m_+ + m_-) n_I U_I^2$ being the dynamical pressure at the subsolar point (m_+ being proton mass, m_- electron mass, n_I number density and U_I velocity of the solar wind) and χ the incident angle of the sun-to-earth velocity with respective to the outward normal to the magnetopause. The pressure balance $\frac{1}{2}\mu_0^{-1}|B|^2 = p_I \cos^2\chi$ (with μ_0 being the magnetic permeability) takes the form of

$$\left(\frac{d\Omega}{dy}\right)^2 = \left(\frac{\Omega_0}{y_0}\right)^2 \tag{5}$$

by virtue of $|B| = d\Omega/d\ell$ and $\cos\chi = dy/d\ell$ (hence $\cos\chi = |B|\, dy/d\Omega$) on the meridional trace of the magnetopause. Here y_0 denotes the y-coordinate of the north neutral point and Ω_0 the magnetic potential at the south neutral point. These two positive-valued parameters, related by $\Omega_0/y_0 = \sqrt{2\mu_0 p_I}$, are to be determined in terms of the two known parameters M_0 and p_I which are indicative of the strengths of the geomagnetic field and incident solar wind.

The crux of the problem is the free boundary on which the y versus Ω condition of equation (5) is to be satisfied. There are two different methodologies to handle the free boundary. Both of them make use of a complex-valued intermediary, $W \equiv U + iV$ or $\rho \equiv r \exp i\theta$, to transform the free boundary in the z-plane into a fixed straight line in the W-plane or a fixed circle in the ρ-plane. In the first approach, extending what was done by Dungey (1961b), the inter-

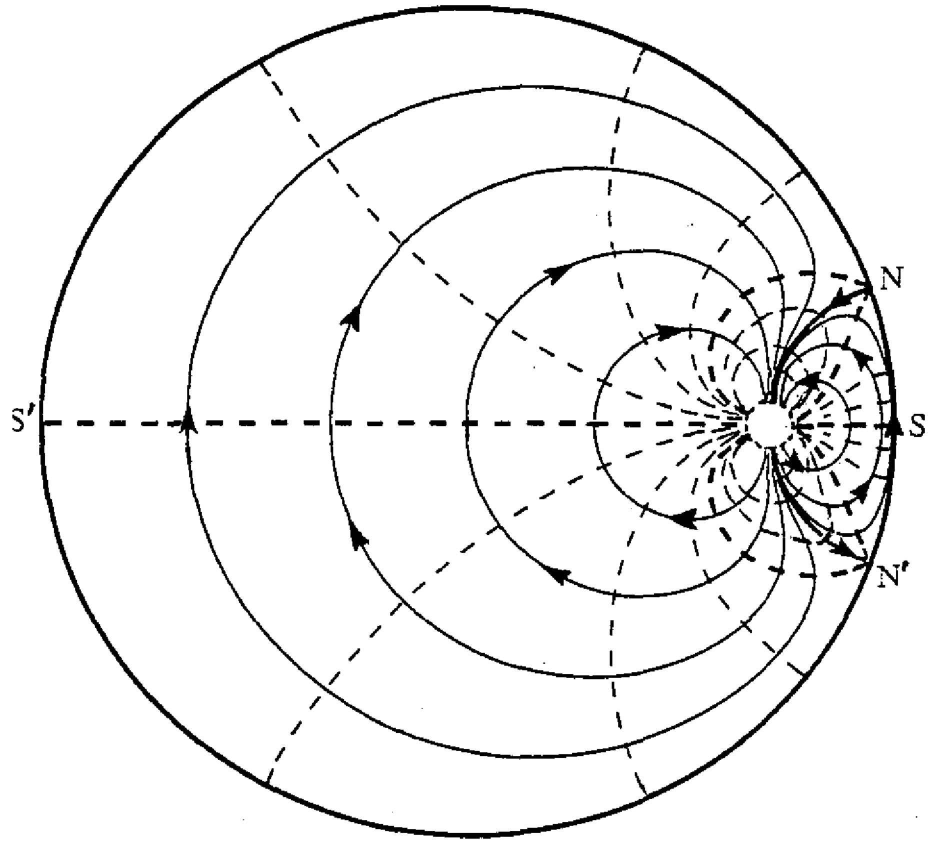

Fig. 1. Topology of two-dimensional magnetospheric cavity. Solid lines indicate field lines and dashed lines equipotential lines. The dotted circle at the origin indicates earth's dipole. North and south neutral points are labeled by N and N'. Subsolar and antisolar points are labeled by S and S'. Subsolar and antisolar segments of magnetopause as well as cusp-to-dipole field lines are indicated by heavy solid lines.

mediary variable W is related to Φ by a conformal mapping between two 'hodograph' planes. In the second approach, used by Hurley (1961), the intermediary variable ζ is related to z by a conformal mapping between two coordinate planes.

3. A SCHWARZ-CHRISTOFFEL TRANSFORMATION

First, we consider the methodology that uses $W \equiv U+iV$ as the intermediary variable. Let $x(\Psi, \Omega)$ and $y(\Psi, \Omega)$ denote the inverses of the pair of functions $\Psi(x, y)$ and $\Omega(x, y)$. Their partial derivatives are related by

$$\begin{bmatrix} \frac{\partial x}{\partial \psi} & \frac{\partial x}{\partial \Omega} \\ \frac{\partial y}{\partial \psi} & \frac{\partial y}{\partial \Omega} \end{bmatrix} \begin{bmatrix} \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} \\ \frac{\partial \Omega}{\partial x} & \frac{\partial \Omega}{\partial y} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{6}$$

The first four partial derivatives, when expressed in terms of the other four, satisfy the Cauchy-Riemann equations:

$$\frac{\partial x}{\partial \Psi} = \frac{\partial y}{\partial \Omega}, \quad \frac{\partial x}{\partial \Omega} = -\frac{\partial y}{\partial \Psi}. \tag{7}$$

Hence, $x(\Psi, \Omega)$ and $y(\Psi, \Omega)$ satisfy Laplace equation:

$$\left(\frac{\partial^2}{\partial \Psi^2} + \frac{\partial^2}{\partial \Omega^2}\right)x = 0, \tag{8}$$

$$\left(\frac{\partial^2}{\partial \Psi^2} + \frac{\partial^2}{\partial \Omega^2}\right)y = 0. \tag{9}$$

In other words, the inverse function $z(\Phi)$ of the analytic function $\Phi(z)$ is analytic too (a well-known theorem in the theory of analytic functions). The function $z(\Phi)$ has a branch cut, from $i\Omega_0$ to $-i\Omega_0$ on the Ω-axis, which interfaces the images of the front and rear segments of magnetopause. See Figure 2. The front segment, which has the subsolar point as its midpoint, and the rear segment, which has the antisolar point as its midpoint, join at the north and south neutral points. The Ω versus y condition on the magnetopause now appears as

$$y|_{\Psi=\pm 0} = \begin{cases} y_0(\Omega/\Omega_0 + 2) & \text{if } \Psi = -0 \text{ and } 0 < \Omega < \Omega_0, \\ -y_0\Omega/\Omega_0 & \text{if } \Psi = +0 \text{ and } -\Omega_0 < \Omega < \Omega_0, \\ y_0(\Omega/\Omega_0 - 2) & \text{if } \Psi = -0 \text{ and } -\Omega_0 < \Omega < 0. \end{cases} \tag{10}$$

Without revealing the methodology, Dungey (1961b) gave the following analytic function

$$z(\Phi) = y_0 \frac{1}{\pi}\left[2 \operatorname{asinh}\frac{\Omega_0}{\Phi} + 2 + i\frac{\Phi + i\Omega_0}{\Omega_0}\log\frac{\Phi + i\Omega_0}{\Phi} - i\frac{\Phi - i\Omega_0}{\Omega_0}\log\frac{\Phi - i\Omega_0}{\Phi}\right] \tag{11}$$

and showed its satisfaction of the condition across the branch cut.

We shall explore a methodology to obtain Dungey's solution by means of a Schwarz-Christoffel transformation (e.g., see Greenberg, 1978). The transformation deforms the whole Φ-plane conformally so that the vertical branch cut in the Φ-plane becomes a horizontal branch

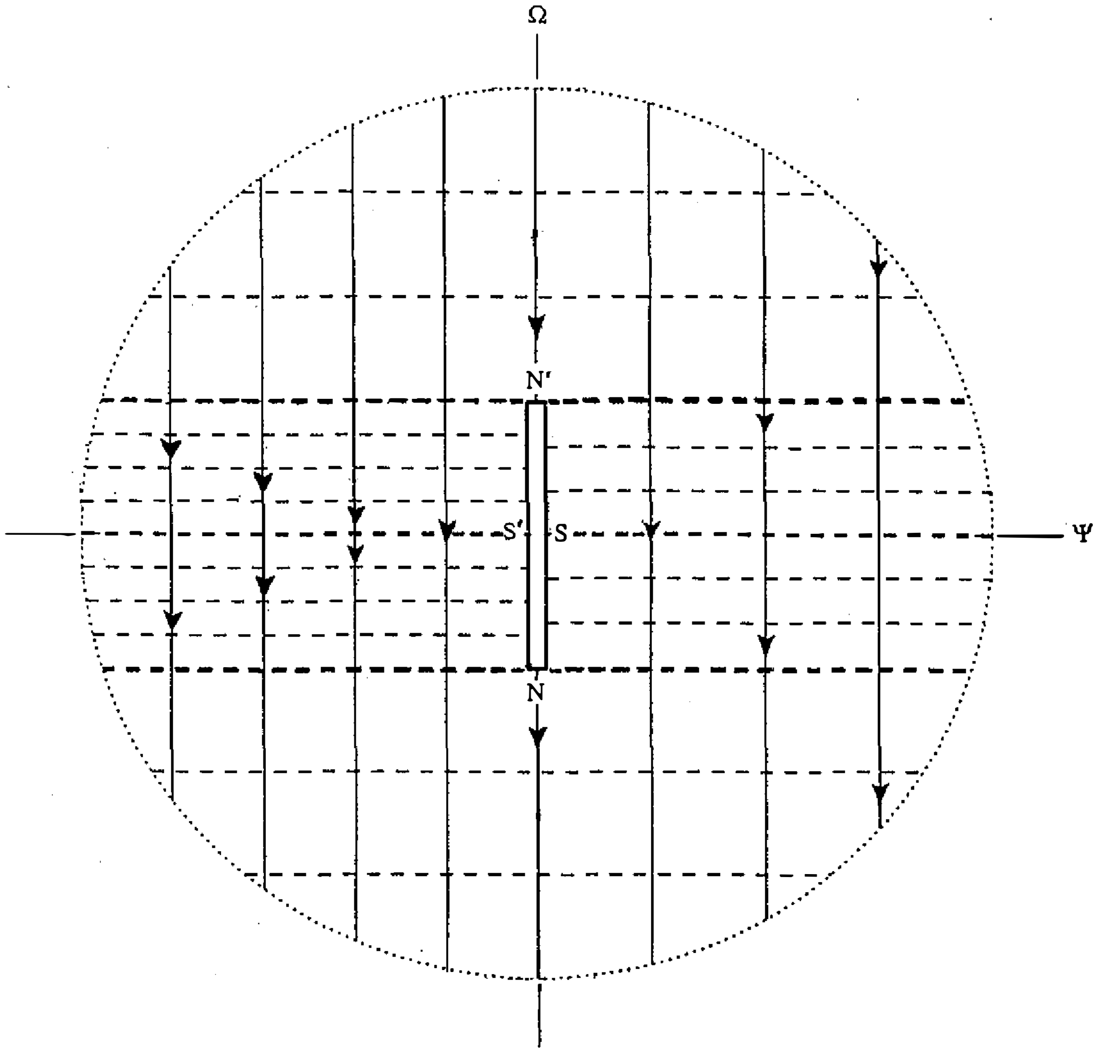

Fig. 2. Mappings of field lines (solid lines), equipotential lines (dashed lines) and the dipole (dotted circle at infinity) in Φ-plane. Mappings of subsolar and antisolar segments of magnetopause (heavy solid lines) are separated by a vertical branch cut in Φ-plane.

cut in the W-plane. See Figure 3. As we go counterclockwise around the boundary of the upper half of the slit Φ-plane, directional turnings of $-\frac{1}{2}\pi$, π and $-\frac{1}{2}\pi$ are encountered at the corners (in the Φ-plane) corresponding to $W=\Omega_0$, 0 and $-\Omega_0$, respectively. Thus, the needed Schwarz-Christoffel transformation is

$$\frac{d\Phi}{dW} = \frac{W}{(W-\Omega_0)^{1/2}(W+\Omega_0)^{1/2}} . \tag{12}$$

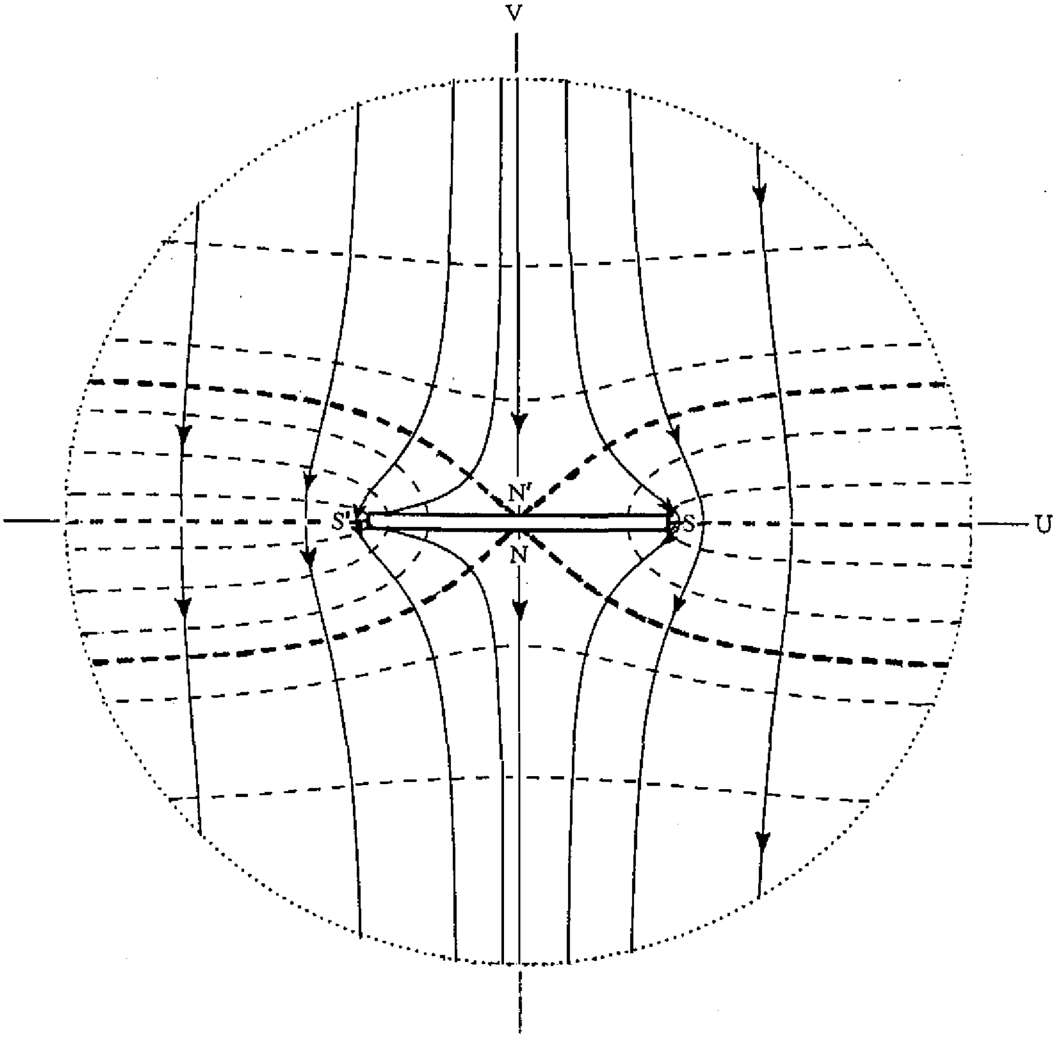

Fig. 3. Mappings of field lines (solid lines), equipotential lines (dashed lines) and the dipole (dotted circle at infinity) in W-plane. Mappings of subsolar and antisolar segments of magnetopause (heavy solid lines) are separated by a horizontal branch cut in W-plane.

Upon integration, we obtain the conformal mapping

$$\Phi = (W - \Omega_0)^{1/2}(W + \Omega_0)^{1/2}, \tag{13}$$

which has the asymptotic behavior of $\Phi \rightarrow W$ in the limit $W \rightarrow \infty$ near the dipole. The inverse mapping is

$$W = (\Phi - i\,\Omega_0)^{1/2}\,(\Phi + i\,\Omega_0)^{1/2}. \tag{14}$$

The mappings of field lines and equipotential lines are orthogonal straight lines in the Φ-plane (see Figure 2) and are orthogonal curves in the W-plane (see Figure 3). The Cauchy-Riemann equations (7) become

$$\frac{\partial x}{\partial U}=\frac{\partial y}{\partial V}, \quad \frac{\partial x}{\partial V}=-\frac{\partial y}{\partial U} \tag{15}$$

and the Laplace equations (8) and (9) become

$$\left(\frac{\partial^2}{\partial U^2}+\frac{\partial^2}{\partial V^2}\right)x=0, \tag{16}$$

$$\left(\frac{\partial^2}{\partial U^2}+\frac{\partial^2}{\partial V^2}\right)y=0. \tag{17}$$

The boundary condition (10) becomes $y=-\dot{y}(U)$ on $V=+0$ of the upper half of the W-plane and $y=\dot{y}(U)$ on $V=-0$ of the lower half plane, with

$$\dot{y}(U)=\begin{cases} y_0\sqrt{1-U^2/\Omega_0^2} & \text{if } 0<U<\Omega_0 \\ y_0(2-\sqrt{1-U^2/\Omega_0^2}) & \text{if } -\Omega_0<U<0 \end{cases} \tag{18}$$

upon the substitution of $\Omega^2+U^2=\Omega_0^2$. The new form of boundary condition supplemented by $y=0$ on the remaining parts of the U-axis as well as $y=0$ at infinity $(U^2+V^2)^{1/2}\rightarrow\infty$ serves as the boundary condition for the Laplace equation (17) for a half plane. This Dirichlet problem for the upper/lower half plane has the solution

$$y(U,V)=\frac{1}{\pi}\int_{-\Omega_0}^{\Omega_0}\dot{y}(\dot{U})\frac{-V}{(U-\dot{U})^2+V^2}d\dot{U} \tag{19}$$

by Poisson's integral formula (e. g., see Greenberg, 1978). Its partial derivatives $\partial y/\partial U$ and $\partial y/\partial V$ provide the values for $-\partial x/\partial V$ and $\partial x/\partial U$, respectively, by virtue of equations (15). From the combination $(\partial x/\partial U)dU+(\partial x/\partial V)dV$ for dx, we obtain

$$x(U,V)=\frac{1}{\pi}\int_{-\Omega_0}^{\Omega_0}\dot{y}(\dot{U})\frac{U-\dot{U}}{(U-\dot{U})^2+V^2}d\dot{U} \tag{20}$$

which tends to 0 when $\sqrt{U^2+V^2}$ approaches ∞ near the dipole. Together, x(U, V)+iy(U, V) can be written

$$z(W) = \frac{1}{\pi}\int_{-\Omega_0}^{\Omega_0} \dot{y}(\dot{U})\,\frac{1}{W-\dot{U}}\,d\dot{U}, \tag{21}$$

which defines an analytic function of W for z. The aymptotic behavior $zW \to \frac{1}{\pi}\int_{-\Omega_0}^{\Omega_0} \dot{y}(\dot{U})d\dot{U}$ near the dipole indicates that the dipole moment is $M_0 = \frac{2}{\pi}y_0\Omega_0$.

To evaluate the integral for z(W), we split $\dot{y}(U)$ into a constant y_0 and an odd function $y_0(1-\sqrt{1-U^2/\Omega_0^2})$. The former yields

$$\int_{-\Omega_0}^{\Omega_0} \frac{1}{W-\dot{U}}\,d\dot{U} = \log\frac{W+\Omega_0}{W-\Omega_0} = 2\ \mathrm{asinh}\frac{\Omega_0}{(W^2-\Omega_0^2)^{1/2}} \tag{22}$$

and the latter yields

$$\int_0^{\Omega_0} (1-\sqrt{1-\dot{U}^2/\Omega_0^2})\left(\frac{1}{W+\dot{U}} - \frac{1}{W-\dot{U}}\right)d\dot{U}$$

$$= 2 - 2\frac{(W^2-\Omega_0^2)^{1/2}}{\Omega_0}\,\mathrm{atan}\frac{\Omega_0}{(W^2-\Omega_0^2)^{1/2}} + \log\frac{W^2-\Omega_0^2}{W^2}. \tag{23}$$

Altogether we obtain

$$z = y_0\frac{1}{\pi}\left[\log\frac{W+\Omega_0}{W-\Omega_0} + 2 + i\,\frac{(W^2-\Omega_0^2)^{1/2}}{\Omega_0}\,\log\frac{(W^2-\Omega_0^2)^{1/2}+i\Omega_0}{(W^2-\Omega_0^2)^{1/2}-i\Omega_0} + \log\frac{W^2-\Omega_0^2}{W^2}\right], \tag{24}$$

which is nothing but Dungey's solution given by equation (11) upon the substitution of equation (14).

4. A CANONICAL CONFIGURATION

Next, we consider the methodology that uses $\zeta \equiv \rho \exp i\theta$ as the intermediary variable. The complex potential

$$\Phi = \frac{M_O}{\zeta} - B_C \zeta . \tag{25}$$

describes a magnetic field $B=(M_O/\zeta^2+B_C)\, d\zeta/dz$, depending on how the intermediary variable ζ is related to the coordinate variable z. Its flux function $\Psi=(M_O/\rho-B_C\rho)\cos\theta$ has a value of 0 on the circle $\rho=\rho_0$ with $\rho_0\equiv(M_O/B_C)^{1/2}$ and also on the polar axis $\theta=\pm\frac{1}{2}\pi$ in the ζ-plane. Its scalar potential $\Omega=-(M_O/\rho+B_C\rho)\sin\theta$ has a sinusoidal variation

$$\Omega = -\Omega_0 \sin\theta, \tag{26}$$

with $\Omega_0\equiv 2M_O/\rho_0$, on that circular boundary. Figure 4 shows the mappings of field lines and equipotential lines, with the mappings of the two neutral points at $\zeta=\rho_0\, e^{\pm i\pi/2}$. In the pedagogic case of ζ=z mapping, the magnetic field M_O/z^2+B_C represents the noon-meridional configuration of a cavity with a spherical magnetopause (cf. Yeh, 1997). This canonical configuration (see Figure 4) has a northward uniform field $\mathbf{1}_y B_C$ to account for the part of magnetic field due to the magnetopause current on the spherical cavity. It is equivalent to the magnetic field due to a pair of line currents at infinity on opposite sides of the dipole, carrying oppositely directed currents of infinite strength proportional to their distance of separation.

A conformal transformation between the ζ-plane and the z-plane will have z as an analytic function of ζ. The two functions $x(\rho, \theta)$ and $y(\rho, \theta)$ satisfy Laplace equations:

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right)x = 0, \tag{27}$$

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2}\right)y = 0. \tag{28}$$

Moreover, the y versus Ω relation (10) on the magnetopause takes the form

$$\ddot{y}(\theta) = \begin{cases} y_0(-2-\sin\theta) & \text{if } -\pi < \theta < -\frac{\pi}{2}, \\ y_0 \sin\theta & \text{if } -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \\ y_0(2-\sin\theta) & \text{if } \frac{\pi}{2} < \theta < \pi. \end{cases} \tag{29}$$

It serves as the boundary condition for equation (28). This Dirichlet problem for a circular domain has the solution

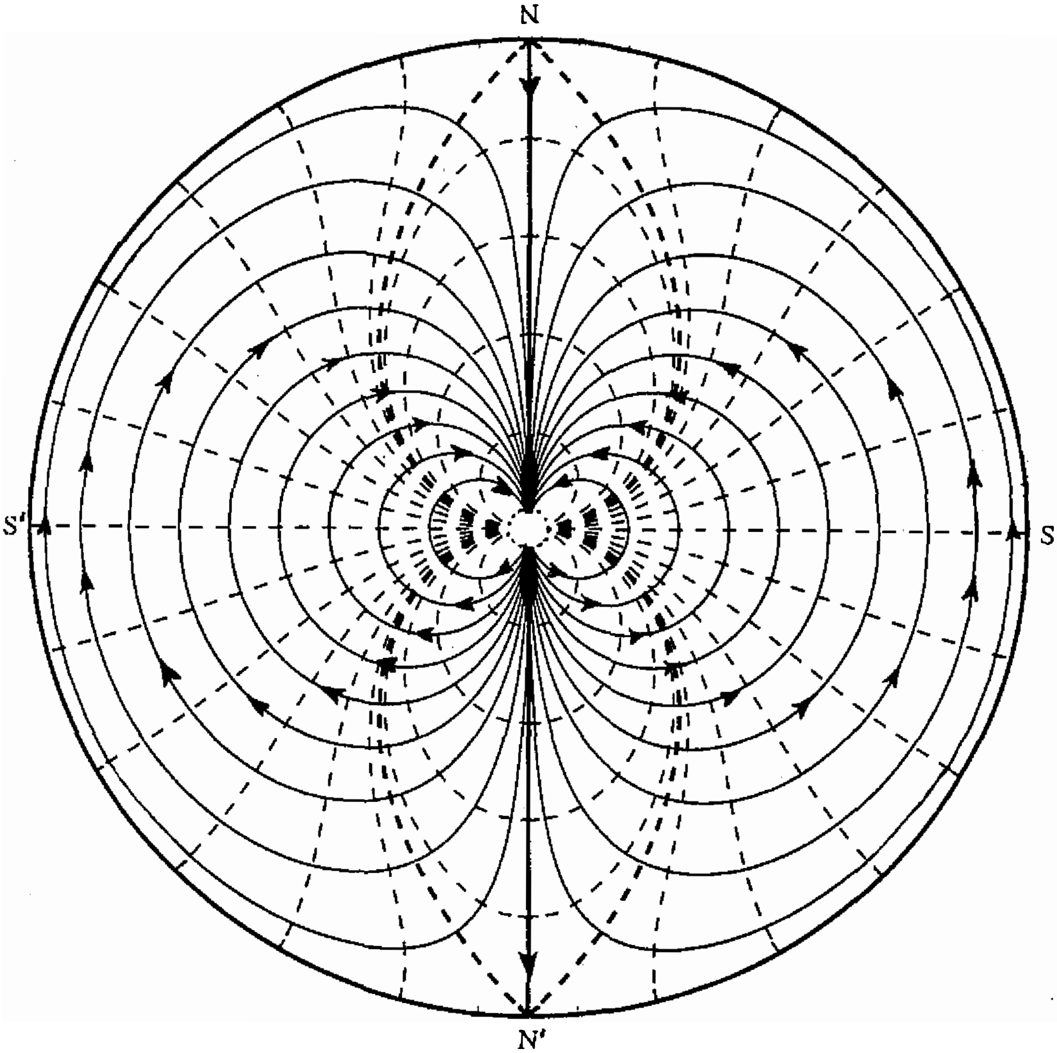

Fig. 4. Mappings of field lines (solid lines), equipotential lines (dashed lines) and the dipole (the dotted circle at the origin) in ζ-plane. Mappings of magnetopause and cusp-to-dipole field lines are indicated by heavy solid lines.

$$y(\rho, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ddot{y}(\dot{\theta}) \frac{\rho_0^2 - \rho^2}{\rho_0^2 - 2\rho_0\rho \cos(\dot{\theta} - \theta) + \rho^2} d\dot{\theta} \tag{30}$$

by Poisson's integral formula (e.g., see Greenberg, 1978). The corresponding $x(\rho, \theta)$ can be found from the Cauchy-Riemann equations

$$\frac{\partial}{\partial\rho}x=\frac{1}{\rho}\frac{\partial}{\partial\theta}y,\qquad \frac{1}{\rho}\frac{\partial}{\partial\theta}x=-\frac{\partial}{\partial\rho}y. \tag{31}$$

It is

$$x(\rho,\theta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\ddot{y}(\dot{\theta})\,\frac{2\rho_0\,\rho\,\sin(\dot{\theta}-\theta)}{\rho_0^2-2\rho_0\,\rho\,\cos(\dot{\theta}-\theta)+\rho^2}\,d\dot{\theta} \tag{32}$$

which tends to 0 when ρ approaches 0 near the dipole. Together, $x(\rho, \theta)+iy(\rho, \theta)$ can be written

$$z(\zeta)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\ddot{y}(\dot{\theta})\,\frac{\rho_0\,e^{i\dot{\theta}}+\zeta}{\rho_0\,e^{i\dot{\theta}}-\zeta}\,i\,d\dot{\theta}\ , \tag{33}$$

which defines an analytic function of ζ for z. When $\ddot{y}(\theta)$ is odd-symmetric, equation (33) becomes

$$z(\zeta)=\frac{1}{\pi}\int_{0}^{\pi}\ddot{y}(\dot{\theta})\,\frac{2\,\rho_0\,\zeta\,\sin\dot{\theta}}{\rho_0^2-2\,\rho_0\,\zeta\cos\dot{\theta}\,+\zeta^2}\,d\dot{\theta}. \tag{34}$$

The asymptotic behavior $z\rightarrow(4y_0/\pi\rho_0)\zeta+O(\zeta^2)$ indicates that we may choose $\rho_0=\frac{4}{\pi}y_0$ to make $z\rightarrow\zeta$ near the dipole. Accordingly, $B_C=(\frac{\pi}{4})^2M_0/y_0^2$ in terms of M_0 and y_0.

To evaluate the integral for $z(\zeta)$, we split the odd function $\ddot{y}(\theta)$ into a constant part and a sinsusoidal part. The former yields

$$\left(-\int_{-\pi}^{-\pi/2}+\int_{\pi/2}^{\pi}\right)2\,\frac{\rho_0\,e^{i\dot{\theta}}+\zeta}{\rho_0\,e^{i\dot{\theta}}-\zeta}\,i\,d\dot{\theta}=-4\log\frac{\rho_0\,e^{-i\pi/2}-\zeta}{\rho_0\,e^{-i\pi}-\zeta}+4\log\frac{\rho_0\,e^{i\pi}-\zeta}{\rho_0\,e^{i\pi/2}-\zeta} \tag{35}$$

and the latter yields

$$\left(-\int_{-\pi}^{-\pi/2}+\int_{-\pi/2}^{\pi/2}-\int_{\pi/2}^{\pi}\right)\sin\dot{\theta}\,\frac{\rho_0\,e^{i\dot{\theta}}+\zeta}{\rho_0\,e^{i\dot{\theta}}-\zeta}\,i\,d\dot{\theta}$$

$$= 4 + i\left(\frac{\rho_0}{\zeta} - \frac{\zeta}{\rho_0}\right)\left(-\log\frac{\rho_0\, e^{-i\pi/2} - \zeta}{\rho_0\, e^{-i\pi} - \zeta} + \log\frac{\rho_0\, e^{i\pi/2} - \zeta}{\rho_0\, e^{-i\pi/2} - \zeta} - \log\frac{\rho_0\, e^{i\pi} - \zeta}{\rho_0\, e^{i\pi/2} - \zeta}\right). \tag{36}$$

Altogether we obtain

$$z = y_0 \frac{1}{\pi}\left[2 + i\left(\frac{\rho_0}{\zeta} - \frac{\zeta}{\rho_0}\right) \log \frac{\rho_0 + i\,\zeta}{\rho_0 - i\,\zeta} + 2 \log\frac{(\rho_0 + \zeta)^2}{\rho_0^2 + \zeta^2}\right], \tag{37}$$

which is nothing but Hurley's solution. It becomes equation (11) upon substitution of

$$\frac{\zeta}{\rho_0} = \frac{(\Phi^2 + \Omega_0^2)^{1/2} - \Phi}{\Omega_0} \tag{38}$$

from equation (25).

The field lines are given by equation (37) with the value of flux function $(M_0/\rho_0)(\rho_0/\rho - \rho/\rho_0)\cos\theta$ kept invariant. In particular, the magnetopause (on which $\zeta = \rho_0\, e^{i\theta}$) is given by

$$z = y_0 \frac{2}{\pi}\left[1 + \sin\theta \log\frac{1 + i\, e^{i\theta}}{1 - i\, e^{i\theta}} + \log\frac{(1 + e^{i\theta})^2}{1 + e^{i2\theta}}\right], \qquad -\pi < \theta < \pi, \tag{39}$$

whose imaginary part is already given by equation (29). The cusp-to-dipole field line (on which $\zeta = i\,\rho_0 \sin\theta$) is given by

$$z = y_0 \frac{1}{\pi}\left[2 + \left(\frac{1}{\sin\theta} + \sin\theta\right) \log\frac{1 - \sin\theta}{1 + \sin\theta} + 2\log\left(1 + i\,\frac{2\sin\theta}{\cos^2\theta}\right)\right], \qquad -\pi < \theta < \pi. \tag{40}$$

The north/south neutral points (at which $\zeta \to \rho_0\, e^{\pm i\pi/2}$) are at $z = \frac{2}{\pi}(1 - \log 2)y_0 \pm i\, y_0$. The subsolar point (at which $\zeta = \rho_0$) is at $z = \frac{2}{\pi}(1 + \log 2)y_0$. The antisolar point (at which $\zeta = \rho_0\, e^{\pm i\pi}$) is at infinity $z = -\infty \pm i\, 2y_0$, being split into two points.

5. DISCUSSION

Figure 5 shows the Dungey-Hurley solution of the Chapman-Ferraro problem with the boundary condition in the form of Ferraro's approximation of pressure balance at the magnetopause. In terms of the earth's dipole moment M_0 and the solar wind's dynamic pressure p_I,

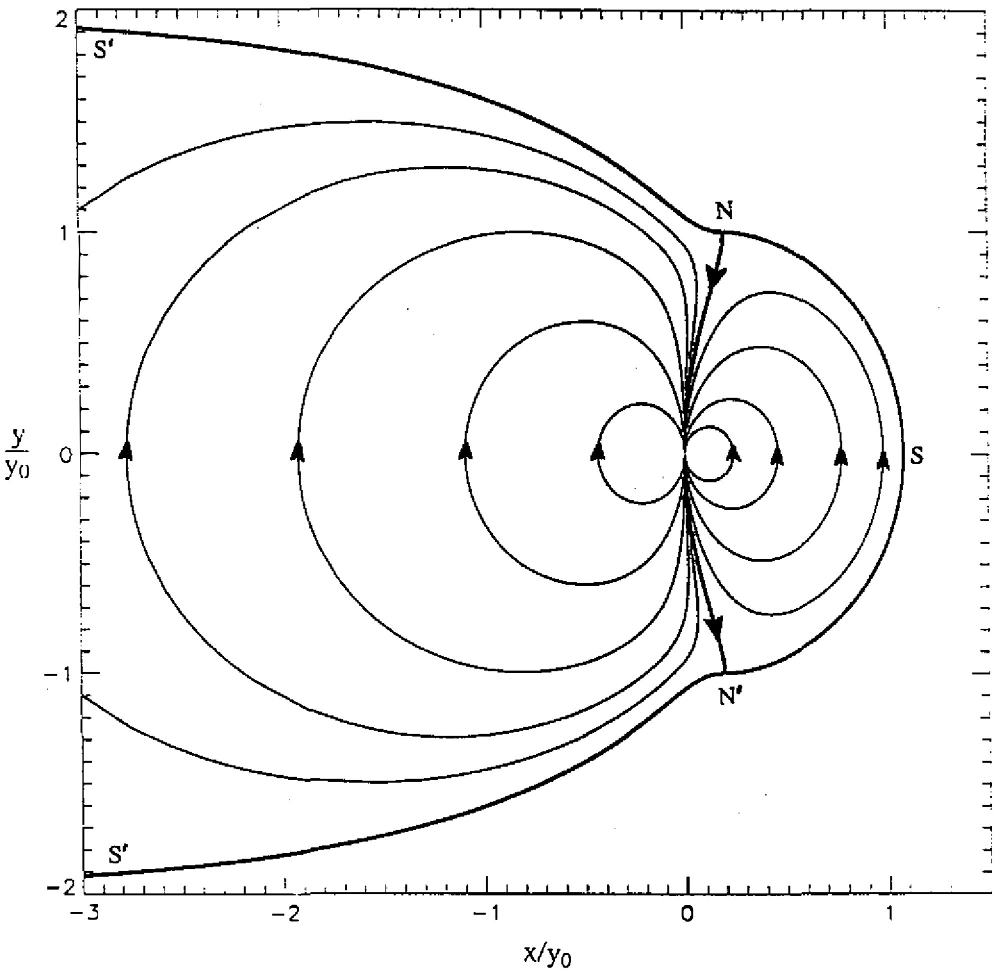

Fig. 5. Dungey-Hurley solution of Chapman-Ferraro problem. Antisolar point is at infinity and magnetopause slope is zero at neutral points.

the two floating parameters have the values $\Omega_0=(\frac{1}{2}\pi M_0\sqrt{2\mu_0 p_I})^{1/2}$ and $y_0=(\frac{1}{2}\pi M_0/\sqrt{2\mu_0 p_I})^{1/2}$. Thus, the two neutral points are at $z=\frac{2}{\pi}(1-\log 2)y_0\pm iy_0$. The subsolar point is at $z=\frac{2}{\pi}(1+\log 2)y_0$ and the antisolar point is split to be at $z=-\infty\pm i\,2y_0$. Receding of the antisolar point to infinity makes the magnetospheric cavity unclosed. At the neutral points, the 90° incident angle necessitated by the vanishing of the magnetic pressure makes the magnetopause slope there parallel to the equatorial plane. These undesired features are inherent in the Ferraro's approximation of pressure balance. They are readily seen in its y versus Ω boundary relation. The first shortcoming of spurious openess is associated with a defect in approximation that the y versus

Ω curve does not terminate at y=0, Ω=0 for the antisolar point. The second shortcoming of zero slope at the neutral points is associated with another defect that $dy/d\Omega$ is not infinite at $y=\pm y_0$, $\Omega=\mp\Omega_0$ to make $|B|\,dy/d\Omega$, which is equal to $\cos\chi$, nonzero at the neutral points.

Indeed, in the instance of the spherical cavity as a canonical closed magnetosphere the boundary condition $y=r_0 \sin\theta$, $\Omega=-\Omega_0 \sin\theta$ indicates that its y versus Ω curve is a closed curve although degenerated into two segments in coincidence. Its magnetopause has a zero slope at the neutral points again because its $dy/d\Omega$ is not infinite there. It is desirable to construct pedagogic models of closed magnetospheres which have a finite cavity and a non-zero magnetopause slope at the neutral points. The general solution presented in this paper for the Chapman-Ferraro problem with arbitrary boundary conditions provides us the needed methodology for obtaining the desirable configuration without the defects in the Dungey-Hurley solution.

An illustrative example is given by the magnetic field

$$B(z) = \frac{M_O}{z^2} + \frac{M_C}{(z - x_C)^2} \tag{41}$$

(see Figure 1 which plots z/x_C for field lines and equipotential lines, with $M_C/M_0=2$). The part of magnetic field due to the magnetopause current is accounted for by an image dipole of southward moment M_C, which is greater than M_0, placed at $z=x_C$ in front of the subsolar point. This magnetic field has two neutral points, located at $z=x_0\pm iy_0$ with $x_0\equiv x_C M_0/(M_0+M_C)$ and $y_0\equiv x_C\sqrt{M_O M_C}/(M_0+M_C)$. Its complex potential

$$\Phi(z) = \frac{M_O}{z} + \frac{M_C}{z - x_C} + \frac{M_C - M_O}{x_C} \tag{42}$$

has the inverse function

$$z(\Phi) = x_C \frac{-M_O/x_C - \frac{1}{2}\Phi + \left(M_O M_C/x_C^2 + \frac{1}{4}\Phi^2\right)^{1/2}}{(M_C - M_O)/x_C - \Phi}, \tag{43}$$

whose two branch points $\Phi=\mp i\Omega_0$ with $\Omega_0\equiv 2\sqrt{M_O M_C}/x_C$ map to the two neutral points. The points $\Phi=\pm 0+i0$ map to the subsolar/antisolar points $z=x_C/(1\pm\sqrt{M_C/M_O})$. The subsolar/antisolar segments (on which $\Psi=\pm 0$) of the magnetopause are given by

$$z\Big|_{\Psi=\pm 0, |\Omega|<\Omega_0} = x_C \frac{-M_O/x_C \pm \sqrt{M_O M_C/x_C^2 - \frac{1}{4}\Omega^2} - i\frac{1}{2}\Omega}{(M_C - M_O)/x_C - i\Omega}$$

$$= x_C\Big[\frac{1}{2} - \frac{1}{2}\frac{(M_C-M_O)/x_C}{(M_O+M_C)/x_C \pm \sqrt{4M_OM_C/x_C^2 - \Omega^2}}$$

$$- i\frac{1}{2}\frac{\Omega}{(M_O+M_C)/x_C \pm \sqrt{4M_OM_C/x_C^2 - \Omega^2}}\Big]. \qquad (44)$$

The antisolar segment peaks at $z=x_C(-M_0 \pm i\sqrt{M_OM_C})/(M_C-M_0)$, with $\Phi=\mp i\Omega_0(M_C-M_0)/(M_0+M_C)$ there. At the neutral points, the magnetopause has the slope $dy/dx=\mp 2/(\sqrt{M_C/M_O} - \sqrt{M_O/M})$, which is obtained from the argument of the complex number $dz/d\Omega$. The cusp-to-dipole field lines (on which $\Psi=0$) are given by

$$z\Big|_{\psi=0,|\Omega|>\Omega_0} = x_C\frac{-M_O/x_C + i\left(\sqrt{\frac{1}{4}\Omega^2 - M_OM_C/x_C^2} - \frac{1}{2}\Omega\right)}{(M_C-M_O)/x_C - i\Omega}. \qquad (45)$$

Now, in terms of the intermediary variable $W=(\Phi^2+\Omega_0^2)^{1/2}$ equation (43) can be written

$$z = x_C\frac{-M_O/x_C + \frac{1}{2}W - \frac{1}{2}(W^2 - \Omega_0^2)^{1/2}}{(M_C-M_O)/x_C - (W^2 - \Omega_0^2)^{1/2}}. \qquad (46)$$

It is nothing but equation (21) with

$$\dot{y}(U) = \frac{1}{2}x_C\frac{\sqrt{\Omega_0^2 - U^2}}{(M_O+M_C)/x_C + U}, \qquad (47)$$

which peaks at $U=-4M_0M_C/(M_0+M_C)x_C$. Likewise, in terms of the intermediary variable $\zeta=\rho_0\Omega_0/[\Phi+(\Phi^2+\Omega_0^2)^{1/2}]$ with $\rho_0\equiv\sqrt{M_O/M_C}\,x_C$ equation (43) can be written

$$z = x_C\frac{\sqrt{M_O/M_C}\rho_0\zeta - \zeta^2}{\rho_0^2 - (\sqrt{M_C/M_O} - \sqrt{M_O/M_C})\rho_0\zeta - \zeta^2}. \qquad (48)$$

It is nothing but equation (34) with

$$\dot{y}(\theta) = x_C \frac{\sin\theta}{\sqrt{M_C/M_O} + \sqrt{M_O/M_C} + 2\cos\theta}, \tag{49}$$

which peaks at $\theta=\mathrm{acos}(-2y_0/x_C)$. In this exemplary cavity, the y versus Ω boundary relation

$$y\Big|_{\psi=\pm 0, |\Omega|<\Omega_0} = \frac{1}{2} x_C \frac{-\Omega}{(M_O+M_C)/x_C \pm \sqrt{4M_O M_C/x_C^2 - \Omega^2}} \tag{50}$$

has $dy/d\Omega$ equal to $\frac{1}{2} x_C^2/(\sqrt{M_O}+\sqrt{M_C})^2$ at the subsolar point, equal to ∞ at the neutral points, equal to 0 at the peak points, and equal to $\frac{1}{2} x_C^2/(\sqrt{M_C}-\sqrt{M_O})^2$ at the antisolar point. The $dy/d\Omega$ slope at the subsolar point is smaller than y_0/Ω_0, which is $\frac{1}{2} x_C^2/(M_o+M_c)$, whereas the slope at the antisolar point is larger. Pressure balance on this cavity requires a non-zero gas pressure at the neutral points on the earthward face to match the dynamic pressure of the solar wind. Moreover, the magnetic pressure on the earthward face of antisolar segment of the boundary is to be matched by that of interplanetary magnetic field because the solar wind does not impinge directly to exert a dynamic pressure there. Accordingly, the interplanetary magnetic field should have an effective value of $2(\sqrt{M_C}-\sqrt{M_O})^2/x_C^2$ at the antisolar point.

Acknowledgments This research was supported by the National Science Council under grant NSC 87-2111-M-008-010-AP8 to the National Central University.

REFERENCES

Chapman, S., and V. C. A. Ferraro, 1931: A new theory of magnetic storms. *Terr. Mag. Atmosph. Elect.*, **36**, 77–97.

Dungey, J. W., 1961a: Interplanetary magnetic field and the auroral zones. *Phys. Rev. Lett.*, **6**, 47–48.

Dungey, J. W., 1961b: The steady state of the Chapman-Ferraro problem in two dimensions. *J. Geophys. Res.*, **66**, 1043–1047.

Ferraro, V. C. A., 1960: An approximate method of estimating the size and shape of the stationary hollow carved out of a neutral ionized stream of corpuscles impinging on the geomagnetic field. *J. Geophys. Res.*, **65** , 3951–3953.

Greenberg, M. D., 1978: Foundations of Applied Mathematics, Prentice-Hall, Englewood

Cliffs, New Jersey, 636pp.

Hurley, J., 1961: Interaction of a streaming plasma with the magnetic field of a two-dimensional dipole. *Phys. Fluids*, **4**, 854–859.

Mead, G. D., and D. B. Beard, 1964: Shape of the geomagnetic field solar wind boundary. *J. Geophys. Res.*, **69**, 1169–1179.

Siscoe, G. L., 1988: The magnetospheric boundary, Physics of Space Plasma (1987). In: T. Chang, G. B. Crew and J. R. Jasperse (Eds.), Scientific Publishers, Cambridge, Massachusetts, 3–78.

Yeh, T., 1997: Magnetospheric structure with a southward interplanetary magnetic field. *J. Geophys. Res.*, **102**, 51–56.